

TRILINEAR APPROACH TO SQUARE FUNCTION AND LOCAL SMOOTHING ESTIMATES FOR THE WAVE OPERATOR

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ABSTRACT. The purpose of this paper is to improve Mockenhaupt's square function estimate and Sogge's local smoothing estimate in \mathbb{R}^3 . For this we use the trilinear approach of S. Lee and A. Vargas for the cone multiplier and some trilinear estimates obtained from the l^2 -decoupling theorem and multilinear restriction theorem.

1. INTRODUCTION

Let $\Gamma = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \tau = |\xi|, 1 \leq \tau \leq 2\}$ be a truncated light cone in \mathbb{R}^3 . For given small $0 < \delta < 1$, let Γ_δ denote the δ -neighborhood of Γ . Let f be a function on \mathbb{R}^3 whose Fourier transform is supported in Γ_δ . To state a square function, we partition Γ_δ into $O(\delta^{-1/2})$ sectors $\Theta = \{(\xi, \tau) \in \Gamma_\delta : \xi/|\xi| \in \theta\}$ corresponding to an arc θ of angular length $O(\delta^{1/2})$ in the unit circle, and let $\mathbf{\Pi}_\delta$ denote the collection of such sectors. We take a collection of Schwartz functions Ξ_Θ so that its Fourier transform $\widehat{\Xi}_\Theta$ is supported on a neighborhood of Θ and $\{\widehat{\Xi}_\Theta\}_{\Theta \in \mathbf{\Pi}_\delta}$ forms a partition of unity of Γ_δ . The square function $S_\delta f$ is defined as

$$S_\delta f = \left(\sum_{\Theta \in \mathbf{\Pi}_\delta} |f_\Theta|^2 \right)^{1/2}$$

where $f_\Theta = f * \Xi_\Theta$. For $1 \leq p \leq \infty$, we say that the square function estimate $\mathcal{SQ}(p \rightarrow p; \alpha)$ holds if the estimate

$$\|f\|_p \leq C_\epsilon \delta^{-\alpha-\epsilon} \|S_\delta f\|_p$$

holds for all $\epsilon > 0$ and all functions f having Fourier support in Γ_δ .

The square function estimate was first considered by Mockenhaupt [11], and the estimate $\mathcal{SQ}(4 \rightarrow 4; 1/8 = 0.125)$ was proved in [11]. It was observed by Bourgain [2] that the exponent α could be less than $1/8$, and Tao and Vargas [13] gave an explicit exponent α by combining bilinear cone restriction estimates with arguments. After that, the sharp bilinear cone restriction estimate was obtained by Wolff [16], and so the estimate $\mathcal{SQ}(4 \rightarrow 4; 5/44 = 0.113\bar{6}\bar{3})$ immediately followed by a theorem in [13]. Garrigós and Seeger [5] have studied l^p decoupling estimates (called Wolff's inequality) for cones, and they further improved the exponent α by combining l^p decoupling estimates with bilinear arguments in [13]. Recently, the sharp l^2 decoupling theorem for the cone was proved by Bourgain and Demeter [3] using the multilinear restriction theorem due to Bennett, Carbery and Tao [1]. So, by results in [5] the estimate $\mathcal{SQ}(4 \rightarrow 4; 3/32 = 0.09375)$ was obtained. Our first result is to make a further progress on the exponent α .

Theorem 1.1. *The estimate $\mathcal{SQ}(4 \rightarrow 4; 1/16 = 0.0625)$ holds.*

The approach to Theorem 1.1 is based on trilinear methods. S. Lee and Vargas [10] already employed a trilinear approach by adapting the arguments of Bourgain and Guth [4], and obtained the sharp estimate $\mathcal{SQ}(3 \rightarrow 3; 0)$. In [10], it was observed that trilinear square function estimates for the cone are essentially equivalent to linear ones. We will use this observation to convert the

linear estimate to trilinear one. For trilinear estimates, the multilinear restriction theorem of Bennet, Carbery and Tao [1] will be utilized too as in [10]. But, to lift L^3 estimate to L^4 estimate we will combine this with the sharp l^2 decoupling theorem due to Bourgain and Demeter [3]. Also, we will adapt the induction-on-scales argument of Bourgain and Demeter [3]. However, since their arguments take advantage of some properties of decoupling norm not derived from the square function, we cannot formulate an iteration as strong as in [3]. Nevertheless, it is enough to obtain Theorem 1.1.

The square function estimate is related to several deep questions in harmonic analysis such as the cone multiplier, local smoothing conjecture and the L^p regularity conjecture for convolution operator with helix. In particular, these conjectures follow from the sharp estimate $\mathcal{SQ}(4 \rightarrow 4; 0)$, see for example [13], [5]. Theorem 1.1 implies the following partial results on these problems.

Corollary 1.2. (i) *If $\alpha > 1/16$ then the local smoothing estimate*

$$\left(\int_1^2 \|e^{it\sqrt{-\Delta}} f\|_{L^4(\mathbb{R}^2)}^4 dt \right)^{1/4} \lesssim \|f\|_{L_\alpha^4(\mathbb{R}^2)}$$

holds, where L_α^p is the L^p -Sobolev space of order α .

(ii) *If $\alpha > 1/16$ then the cone multiplier operator T_α defined by $\widehat{T_\alpha f}(\xi, \tau) = \rho(\tau)(1 - |\xi|^2/\tau^2)_+^\alpha \widehat{f}(\xi)$ is bounded on L^4 , where ρ is a bump function on $[1, 2]$.*

(iii) *If $\alpha < 5/24$ then the convolution operator T defined by*

$$Tf(x) = \int f(x_1 - \cos t, x_2 - \sin t, x_3 - t) \phi(t) dt$$

maps L^4 to L_α^4 , where ϕ is a bump function.

The proof is well known, and we will not reproduce here, see for example [13]. For other related problems, see [5], [3].

Next, we are further concerned with $L_\alpha^p \rightarrow L^q$ type local smoothing estimates

$$\left(\int_1^2 \|e^{it\sqrt{-\Delta}} f\|_{L^q(\mathbb{R}^2)}^q dt \right)^{1/q} \leq C_{p,q,\alpha} \|f\|_{L_\alpha^p(\mathbb{R}^2)}. \quad (1.1)$$

It has been conjectured (see [13], [12]) that this local smoothing estimate holds if

$$1 \leq p \leq q \leq \infty, \quad \frac{1}{p} + \frac{3}{q} = 1, \quad \alpha > \frac{1}{p} - \frac{3}{q} + \frac{1}{2}. \quad (1.2)$$

(When $p = q = 4$ and $\alpha = 0$, it is known that the local smoothing estimate does not hold. But, for $q > 4$, $\frac{1}{p} + \frac{3}{q} = 1$ and $\alpha = \frac{1}{p} - \frac{3}{q} + \frac{1}{2}$, it is not known whether the local smoothing estimate holds or not.)

The critical $L_\alpha^4 \rightarrow L^4$ estimate has been considered in Corollary 1.2. Now we continue to study a sharp $L_\alpha^p \rightarrow L^q$ estimate when $p < q$. From Strichartz' estimate $L_{1/2}^2 \rightarrow L^6$, this conjecture follows for $q \geq 6$. Schlag and Sogge [12] first improved this to $q \geq 5$, and Tao and Vargas [13] made further progress by using bilinear approach. By the bilinear cone restriction estimate due to Wolff [16] and the results in [13], the conjecture was improved to $q \geq 14/3 = 4.\bar{6}$, and the ϵ -loss of α was removed by S. Lee [9]. Our second result is to obtain an improved sharp local smoothing estimate.

Theorem 1.3. *The estimate (1.1) holds for $q \geq 30/7 = 4.\dot{2}8571\dot{4}$ and (1.2).*

Theorem 1.3 will be proved through trilinear approach too. The proof is simpler than Theorem 1.1. We will reduce this linear estimate to a trilinear one, and the desired trilinear estimate will be obtained from interpolating between two trilinear estimates deduced from the multilinear restriction theorem [1] and the l^2 decoupling theorem [3].

Throughout this paper, we write $A \lesssim B$ or $A = O(B)$ if $A \leq CB$ for some constant $C > 0$ which may depend on p, q but not on δ, R and N , and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The constants $C, C_\varepsilon, C_\epsilon, C_{\epsilon_1}$ and the implicit constants in \lesssim and \sim will be adjusted numerous times throughout the paper. For any finite set A , we use $\#A$ to denote its cardinality, and if A is a measurable set, we use $|A|$ to denote its Lebesgue measure. If R is a rectangular box or an ellipsoid and k is a positive real number, we use kR to denote the k -dilation of R with center of dilation at the center of R .

2. REDUCTION TO A TRILINEAR ESTIMATE

In this section, we will show that the linear square function estimate is equivalent to a trilinear one. The arguments of this section are a small modification of [10]. In fact, Proposition 2.2 is obtained by replacing L^3 -norm with L^p -norm in [10].

We begin with the statement of a trilinear square function estimate. Let $\Omega_1, \Omega_2, \Omega_3 \subset S^1$ be arcs with $|\Omega_i| \sim 1$. We say that $\Gamma^{\Omega_1}, \Gamma^{\Omega_2}, \Gamma^{\Omega_3}$ are ν -transverse if for any unit normal vector n_i at Γ^{Ω_i} , $i = 1, 2, 3$, the parallelepiped formed by n_1, n_2, n_3 has volume $\geq \nu$. Let us use the notation $\mathcal{SQ}(p \times p \times p \rightarrow p; \alpha)$ if one has the trilinear square function estimate

$$\left\| \left(\prod_{i=1}^3 |f_i| \right)^{1/3} \right\|_p \leq C_{\nu, \epsilon} \delta^{-\alpha - \epsilon} \left(\prod_{i=1}^3 \|S_\delta f_i\|_p \right)^{1/3}$$

for all $\epsilon > 0$ and all f_i with $\text{supp } \hat{f}_i \subset \Gamma_\delta^{\Omega_i}$, where $\Omega_1, \Omega_2, \Omega_3$ are any arcs with size $O(1)$ such that $\Gamma^{\Omega_1}, \Gamma^{\Omega_2}, \Gamma^{\Omega_3}$ are ν -transverse.

It is easy to see that $\mathcal{SQ}(p \rightarrow p; \alpha)$ means $\mathcal{SQ}(p \times p \times p \rightarrow p; \alpha)$ by Hölder's inequality. So, we will show that the inverse is true. Let $1 > \gamma_1 > \gamma_2 > 0$ be small positive numbers. For $j = 1, 2$, we define $\mathbf{\Omega}(\gamma_j)$ to be a family of $O(\gamma_j^{-1})$ arcs of length $O(\gamma_j)$ covering the unit circle with finite overlap. For an arc $\Omega \subset S^1$ we define a sector Γ^Ω and a δ -fattened sector Γ_δ^Ω by

$$\Gamma^\Omega = \{(\xi, \tau) \in \Gamma : \xi/|\xi| \in \Omega\}, \quad \Gamma_\delta^\Omega = \{(\xi, \tau) \in \Gamma_\delta : \xi/|\xi| \in \Omega\},$$

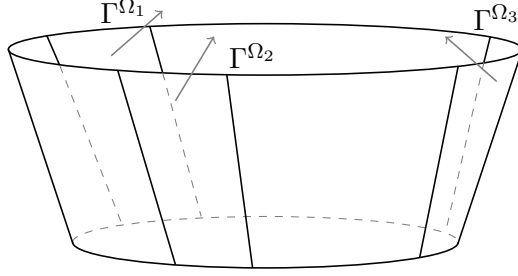
and take a Schwartz function Ξ_Ω whose Fourier transform is a bump function supported on a neighborhood of Γ_δ^Ω . The following is due to S. Lee and Vargas [10]*equation (23).

Lemma 2.1. *Suppose that f has Fourier support in Γ_δ . Then for any $x \in \mathbb{R}^3$,*

$$|f(x)| \lesssim \max_{\Omega \in \mathbf{\Omega}(\gamma_1)} |f_\Omega(x)| + \gamma_1^{-1} \max_{\Omega \in \mathbf{\Omega}(\gamma_2)} |f_\Omega(x)| + \gamma_2^{-50} \max_{\substack{\Omega_1, \Omega_2, \Omega_3 \in \mathbf{\Omega}(\gamma_2): \\ \text{dist}(\Omega_i, \Omega_j) \geq \gamma_2, i \neq j}} \left(\prod_{i=1}^3 |f_{\Omega_i}(x)| \right)^{1/3} \quad (2.1)$$

where $f_\Omega = f * \Xi_\Omega$.

To obtain the above lemma, they adapted the arguments of Bourgain and Guth [4] who improved the restriction conjecture by using a multilinear approach.



We note that $\Gamma^{\Omega_1}, \Gamma^{\Omega_2}, \Gamma^{\Omega_3}$ are ν -transverse if and only if $\Omega_1, \Omega_2, \Omega_3$ are mutually separated by a distance $\gtrsim \nu^{1/3}$, see [10]. Using Lemma 2.1 we can establish the following relation between the linear and trilinear square function estimates.

Proposition 2.2. *Let $p \geq 2$ and $\alpha \geq 0$. Suppose that $\mathcal{SQ}(p \times p \times p \rightarrow p; \alpha)$ holds. Then $\mathcal{SQ}(p \rightarrow p; \alpha)$ is valid.*

Proof. We will use induction arguments. Let $\epsilon > 0$ be given. We assume that $\beta \geq 0$ is the best exponent for which

$$\|f\|_p \leq C\delta^{-\beta-\epsilon}\|S_\delta f\|_p \quad (2.2)$$

holds for all f with $\text{supp } \hat{f} \subset \Gamma_\delta$. It suffices to show that for any small $0 < \epsilon_1 < 1$,

$$\beta \leq \alpha + O(\epsilon_1) + \log_{1/\delta} C_{\epsilon, \epsilon_1}. \quad (2.3)$$

Here, we may take $\epsilon_1 = \epsilon$. The ϵ came from the induction hypothesis, and ϵ_1 is related to the transversality of trilinear estimates below.

We may assume that $\delta > 0$ is sufficiently small, say $0 < \delta \leq \delta_0$, because the desired estimate is trivially obtained, otherwise. Let $1 > \gamma_1 > \gamma_2 \geq \delta_0^{\epsilon_1/2}$, and let γ_1, γ_2 be dyadic multiples of $\delta^{1/2}$. Later we will choose γ_1, γ_2 and δ_0 . By Lemma 2.1 and embedding $l^p \subset l^\infty$,

$$\begin{aligned} \|f\|_p^p &\lesssim \sum_{\Omega_1 \in \Omega(\gamma_1)} \|f_{\Omega_1}\|_p^p + \gamma_1^{-p} \sum_{\Omega_2 \in \Omega(\gamma_2)} \|f_{\Omega_2}\|_p^p \\ &\quad + \gamma_2^{-50p} \sum_{\substack{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_2): \\ \text{dist}(\Omega_i, \Omega_j) \geq \gamma_2, i \neq j}} \left\| \left(\prod_{i=1}^3 |f_{\Omega_i}| \right)^{1/3} \right\|_p^p, \end{aligned} \quad (2.4)$$

where Ω_j is taken such that if θ intersects Ω_j then $\theta \subset \Omega_j$ for $j = 1, 2$.

Consider the first and second summation in the right-hand side of (2.4). For convenience we denote $\Omega = \Omega_j$ and $\gamma = \gamma_j$. Using rescaling we will show

$$\|f_\Omega\|_p \leq C_\epsilon (\delta/\gamma^2)^{-\beta-\epsilon} \|S_\delta f_\Omega\|_p. \quad (2.5)$$

By rotating the unit circle we may assume that Ω is centered at $(1, 0)$. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation so that

$$T(e_1, 1) = (e_1, 1), \quad T(-e_1, 1) = \gamma^2(-e_1, 1), \quad T(e_2, 0) = \gamma(e_2, 0)$$

where $\{e_1, e_2\}$ is a standard basis in \mathbb{R}^2 . Then $\hat{f}_\Omega \circ T$ is supported in Γ_{δ/γ^2} . From the equation $\widehat{f_\Omega \circ T^{-t}} = |\det T| \widehat{f_\Omega \circ T}$, it follows that $\widehat{f_\Omega \circ T^{-t}}$ has support in Γ_{δ/γ^2} where T^{-t} is the inverse transpose of T . Since $\gamma \geq \delta^{1/2}$, by (2.2) it follows that

$$\|f_\Omega \circ T^{-t}\|_p \lesssim (\delta/\gamma^2)^{-\beta-\epsilon} \|S_{\delta/\gamma^2}(f_\Omega \circ T^{-t})\|_p. \quad (2.6)$$

By definition,

$$S_{\delta/\gamma^2}(f_{\Omega} \circ T^{-t}) = \left(\sum_{\Upsilon \in \Pi_{\delta/\gamma^2}} |(f_{\Omega} \circ T^{-t}) * \Xi_{\Upsilon}|^2 \right)^{1/2}.$$

From $\hat{\Xi}_{\Upsilon} \circ T^{-1} = \hat{\Xi}_{T(\Upsilon)}$, it follows that $((f_{\Omega} \circ T^{-t}) * \Xi_{\Upsilon})^{\wedge} = |\det T|(\hat{f}_{\Omega} \circ T)\hat{\Xi}_{\Upsilon} = |\det T|(\hat{f}_{\Omega} \hat{\Xi}_{T(\Upsilon)}) \circ T$. Thus, by taking the inverse Fourier transform,

$$(f_{\Omega} \circ T^{-t}) * \Xi_{\Upsilon} = (f_{\Omega} * \Xi_{T(\Upsilon)}) \circ T^{-t}.$$

Since $f_{\Omega} * \Xi_{T(\Upsilon)}$ has Fourier support in $T(\Upsilon)$ which is a sector of size $1 \times \delta \times C\delta^{1/2}$ in Γ_{δ} , we have

$$S_{\delta/\gamma^2}(f_{\Omega} \circ T^{-t}) = \left(\sum_{\Upsilon \in \Pi_{\delta/\gamma^2}} |(f_{\Omega} * \Xi_{T(\Upsilon)}) \circ T^{-t}|^2 \right)^{1/2} = (S_{\delta} f_{\Omega}) \circ T^{-t}.$$

We substitute this in (2.6) and remove T^{-t} by changing variables. Then we obtain (2.5).

By (2.5) we have

$$\sum_{\Omega \in \Omega(\gamma)} \|f_{\Omega}\|_p^p \leq C_{\epsilon}(\delta/\gamma^2)^{-p\beta-p\epsilon} \sum_{\Omega \in \Omega(\gamma)} \|S_{\delta} f_{\Omega}\|_p^p.$$

Since we can decompose $f_{\Omega} = \sum_{\Theta \in \Pi_{\delta}: \theta \subset \Omega} f * \Xi_{\Theta}$, we have that for $p \geq 2$,

$$\begin{aligned} \sum_{\Omega \in \Omega(\gamma)} \|S_{\delta} f_{\Omega}\|_p^p &= \sum_{\Omega \in \Omega(\gamma)} \int \left(\sum_{\Theta \in \Pi_{\delta}: \theta \subset \Omega} |f * \Xi_{\Theta}|^2 \right)^{p/2} \\ &\leq \int \left(\sum_{\Omega \in \Omega(\gamma)} \sum_{\Theta \in \Pi_{\delta}: \theta \subset \Omega} |f * \Xi_{\Theta}|^2 \right)^{p/2} \\ &\leq \|S_{\delta} f\|_p^p. \end{aligned}$$

Inserting this into the previous estimate, we obtain

$$\sum_{\Omega \in \Omega(\gamma)} \|f_{\Omega}\|_p^p \leq C_{\epsilon}(\delta/\gamma^2)^{-p\beta-p\epsilon} \|S_{\delta} f\|_p^p. \quad (2.7)$$

Consider the trilinear part in (2.4). By applying $\mathcal{SQ}(p \times p \times p \rightarrow p; \alpha)$,

$$\sum_{\substack{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_2): \\ \text{dist}(\Omega_i, \Omega_j) \geq \gamma_2, i \neq j}} \left\| \left(\prod_{i=1}^3 |f_{\Omega_i}| \right)^{1/3} \right\|_p^p \leq C_{\epsilon, \gamma_2} \gamma_2^{-3} \delta^{-p\alpha-p\epsilon} \|S_{\delta} f\|_p^p. \quad (2.8)$$

We substitute (2.7) and (2.8) in (2.4). Then,

$$\|f\|_p \leq (C_{\epsilon} \gamma_1^{2(\beta+\epsilon)} \delta^{-\beta-\epsilon} + C_{\epsilon} \gamma_1^{-1} \gamma_2^{2(\beta+\epsilon)} \delta^{-\beta-\epsilon} + C_{\epsilon, \gamma_2} \gamma_2^{-60} \delta^{-\alpha-\epsilon}) \|S_{\delta} f\|_p.$$

So, by the assumption for β ,

$$\delta^{-\beta} \leq (C_{\epsilon} \gamma_1^{2(\beta+\epsilon)} + C_{\epsilon} \gamma_1^{-1} \gamma_2^{2(\beta+\epsilon)}) \delta^{-\beta} + C_{\epsilon, \gamma_2} \gamma_2^{-60} \delta^{-\alpha}.$$

We now choose γ_1, γ_2 and δ_0 so that $C_{\epsilon} \gamma_1^{2(\beta+\epsilon)} \leq 1/4$, $C_{\epsilon} \gamma_1^{-1} \gamma_2^{2(\beta+\epsilon)} \leq 1/4$ and $1 > \gamma_1 > \gamma_2 \geq \delta_0^{\epsilon_1/2}$. Then $\delta^{-\beta} \leq C_{\epsilon, \gamma_2} \gamma_2^{-60} \delta^{-\alpha} \leq C_{\epsilon, \epsilon_1} \delta^{-60\epsilon_1-\alpha}$, which means (2.3). \square

3. DECOUPLING NORM

In this section, we will show that the decoupling norm essentially satisfies the reverse Hölder inequality, and apply this to the interpolation between decoupling estimates. This section will be obtained by modifying [3]*section 3. For further discussion for decoupling, see [15], [8], [7], [6].

Let f be a function having Fourier support in Γ_δ . For such functions, the norm $\|\cdot\|_{p,\delta}$, $2 \leq p \leq \infty$ is defined by

$$\|f\|_{p,\delta} := \left(\sum_{\Theta \in \mathbf{\Pi}_\delta} \|f_\Theta\|_p^2 \right)^{1/2}.$$

It is easy to see that if m is a positive integer then $\|f\|_{p,m\delta} \leq C_m \|f\|_{p,\delta}$ by Minkowski's inequality.

We first introduce a wave packet decomposition, which is a fundamental tool for studying Fourier restriction type problems. To decompose f both in frequency space and in spatial space, we first define standard bump functions. Let $\phi(x) := (1 + |x|^2)^{-M/2}$ where M is a sufficiently large exponent. Let $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a nonnegative Schwartz function such that ψ is strictly positive in the unit ball $B(0,1)$, Fourier supported in a ball $B(0,1/4)$ and $\sum_{k \in \mathbb{Z}^3} \psi(x-k) = 1$. For an ellipsoid E , we define a_E to be an affine map from the unit ball $B(0,1)$ to E . Let $\phi_E = \phi \circ a_E^{-1}$ and $\psi_E = \psi \circ a_E^{-1}$.

Lemma 3.1. *Suppose that f is Fourier supported in Γ_δ . Then there exists a decomposition*

$$f(x) = \sum_{\Theta \in \mathbf{\Pi}_\delta} \sum_{\pi \in \mathbf{P}_\Theta} h_\pi f_\pi(x), \quad (3.1)$$

where $\mathbf{P}_\Theta = \mathbf{P}_\Theta(f)$ is a family of separated rectangles π of size $\delta^{-1} \times \delta^{-1/2} \times 1$ with its dual $\pi^* = \Theta$, such that the coefficients $h_\pi > 0$ obey

$$\left(\sum_{\Theta \in \mathbf{\Pi}_\delta} \left(\sum_{\pi \in \mathbf{P}_\Theta} |\pi| h_\pi^p \right)^{2/p} \right)^{1/2} = \|f\|_{p,\delta}, \quad (3.2)$$

$$\text{supp } \hat{f}_\pi \subset 4\Theta, \quad (3.3)$$

and

$$|f_\pi(x)| \lesssim \phi_\pi(x). \quad (3.4)$$

Proof. For each $\Theta \in \mathbf{\Pi}_\delta$, we partition \mathbb{R}^3 into the dual rectangles π of Θ . For each π , we define a coefficient h_π and a function f_π by

$$h_\pi = \left(\frac{1}{|\pi|} \int |f_\Theta|^p \psi_\pi \right)^{1/p}, \quad f_\pi(x) = h_\pi^{-1} \psi_\pi(x) f_\Theta(x).$$

Then, (3.3) immediately follows, and some direct calculating gives (3.1). Also, by the definition of h_π ,

$$\sum_{\pi \in \mathbf{P}_\Theta} |\pi| h_\pi^p = \sum_{\pi \in \mathbf{P}_\Theta} \int |f_\Theta|^p \psi_\pi = \|f_\Theta\|_p^p.$$

From this, (3.2) follows. Lastly, by Bernstein's inequality,

$$|\psi_\pi(x) f_\Theta(x)| \lesssim |\Theta|^{1/p} \|\psi_\pi f_\Theta\|_p = h_\pi,$$

so we have $|f_\pi(x)| \lesssim |\psi_\pi(x)|$. This implies (3.4). \square

Now we study the reverse Hölder inequality for the decoupling norm. We say that f is a *balanced function* if f is a function of the form (3.1) with $h_\pi = 1$ and satisfying (3.3), (3.4) and a property that for any $\Theta, \Theta' \in \mathbf{\Pi}_\delta$, the nonempty $\mathbf{P}_\Theta(f), \mathbf{P}_{\Theta'}(f)$ have comparable cardinality.

Lemma 3.2. *Suppose that $2 \leq p, q, r \leq \infty$ and that for some $\theta \in (0, 1)$,*

$$\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{p}.$$

Then

$$\|f\|_{r,\delta} \sim \|f\|_{q,\delta}^{1-\theta} \|f\|_{p,\delta}^{\theta},$$

for all balanced function f .

Proof. Since f is a balanced function, there is a number $\kappa > 0$ such that every nonempty $\mathbf{P}_{\Theta}(f)$ has cardinality comparable to κ . Let ν be the number of nonempty $\mathbf{P}_{\Theta}(f)$. Then by (3.2), one has

$$\|f\|_{r,\delta} \sim \nu^{1/2} \kappa^{1/r} |\pi|^{1/r} = \nu^{\frac{1-\theta}{2}} \kappa^{\frac{1-\theta}{q}} |\pi|^{\frac{1-\theta}{q}} \nu^{\frac{\theta}{2}} \kappa^{\frac{\theta}{p}} |\pi|^{\frac{\theta}{p}} \sim \|f\|_{q,\delta}^{1-\theta} \|f\|_{p,\delta}^{\theta}.$$

□

As an application we have the following interpolation lemma.

Lemma 3.3. *Assume that*

$$\|f\|_{q_1} \leq A_1 \|f\|_{p_1,\delta}, \quad \|f\|_{q_2} \leq A_2 \|f\|_{p_2,\delta} \quad (3.5)$$

for all f with $\text{supp } \hat{f} \subset \Gamma_{\delta}$. Suppose that for some $\theta \in (0, 1)$,

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$

and $2 \leq p \leq q \leq \infty$. Then

$$\|f\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^{\theta} \|f\|_{p,\delta} \quad (3.6)$$

for all f with $\text{supp } \hat{f} \subset \Gamma_{\delta}$ and all $\varepsilon > 0$.

It is worth pointing out that the above lemma cannot be obtained from the usual interpolation theorem because f has a Fourier support condition. To avoid this obstruction we can use a method in [5] and [6]. But, since the method utilizes a Besicovitch-type maximal operator, it is hard to obtain optimal estimates when applied to high dimensional cases. The arguments in this section works well in general dimensions. So, it strongly seems that Lemma 3.3 and Lemma 3.4 are generalized to high dimensional ones.

Proof. For localization we decompose $f = \sum_{k \in \delta^{-1}\mathbb{Z}^3} \psi_k f$ where $\psi_k := \psi(\delta(x - k))$. Then,

$$\|f\|_q^q \leq \sum_{k' \in \delta^{-1}\mathbb{Z}^3} \left\| \sum_{k \in \delta^{-1}\mathbb{Z}^3} \psi_k f \right\|_{L^q(B(k', 2\delta^{-1}))}^q$$

Since ψ_k has rapid decay outside $B(k, \delta^{-1-\varepsilon})$, we have that if $x \in B(k', 2\delta^{-1})$ then

$$\left| \sum_{k \in \delta^{-1}\mathbb{Z}^3 \setminus B(k', 2\delta^{-1-\varepsilon})} \psi_k(x) \right| \leq C_K \delta^K$$

for all $K > 0$. Using this and a rough estimate $\|f\|_q \lesssim \delta^{-C} \|f\|_{p,\delta}$, we have that for any $\varepsilon > 0$ and $K > 0$,

$$\|f\|_q^q \leq \sum_{k'} \left\| \sum_{k \sim k'} \psi_k f \right\|_{L^q(B(k', 2\delta^{-1}))}^q + C_K \delta^K \|f\|_{p,\delta}^q,$$

where $k \sim k'$ means that $k \in B(k', 2\delta^{-1-\varepsilon}) \cap \delta^{-1}\mathbb{Z}^3$. Since the number of $k \in \delta^{-1}\mathbb{Z}^3$ contained in $B(k', 2\delta^{-1-\varepsilon})$ is $O(\delta^{3\varepsilon})$, we have

$$\begin{aligned} \|f\|_q^q &\lesssim \delta^{-3\varepsilon q} \sum_{k'} \sum_{k \sim k'} \|\psi_k f\|_{L^q(B(k', 2\delta^{-1}))}^q + C_K \delta^K \|f\|_{p,\delta} \\ &\lesssim \delta^{-3\varepsilon q} \sum_{k'} \sum_{k \sim k'} \|\psi_k f\|_q^q + C_K \delta^K \|f\|_{p,\delta} \\ &\lesssim \delta^{-3\varepsilon q - 3\varepsilon} \sum_k \|\psi_k f\|_q^q + C_K \delta^K \|f\|_{p,\delta}. \end{aligned}$$

Since $p \leq q$, we have that for any $\varepsilon > 0$ and any $K > 0$,

$$\|f\|_q \lesssim \delta^{-C\varepsilon} \left(\sum_k \|\psi_k f\|_q^p \right)^{1/p} + C_K \delta^K \|f\|_{p,\delta}.$$

On the other hands, by Minkowski's inequality it follows that

$$\left(\sum_k \|\psi_k f\|_{p,2\delta}^p \right)^{1/p} \leq \|f\|_{p,2\delta} \lesssim \|f\|_{p,\delta}.$$

Thus, by the above two estimates the proof of (3.6) is reduced to showing

$$\|\psi_k f\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta \|\psi_k f\|_{p,2\delta}.$$

By translation invariance it is enough to consider $\psi_0 f$. Let $g := \psi_0 f$. By normalization we may assume that $\|g\|_{p,2\delta} = 1$. Then it is reduced to showing

$$\|g\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta. \quad (3.7)$$

Since ψ_0 has fast decay outside $B(0, C\delta^{-1})$, we have $\|g\|_q \leq \|g\|_{L^q(B(0, \delta^{-1-\varepsilon}))} + C_K \delta^K$ for all $\varepsilon > 0$ and $K > 0$. Since ψ_0 has Fourier support in $B(0, \delta/2)$, \widehat{g} is supported in $\Gamma_{2\delta}$. By Lemma 3.1, it is decomposed into

$$g(x) = \sum_{\Theta \in \Pi_{2\delta}} \sum_{\pi \in \mathbf{P}_\Theta} h_\pi g_\pi(x).$$

We first remove some minor π 's. By (3.4), we can eliminate π that is disjoint from $B(0, C\delta^{-1-\varepsilon})$. Let $\mathring{\mathbf{P}}$ be the collection of π intersecting $B(0, C\delta^{-1-\varepsilon})$. Then $\#\mathring{\mathbf{P}} \lesssim \delta^{-3/2-3\varepsilon}$. The rectangles π with $h_\pi = O(\delta^{500})$ can be also eliminated since

$$\left\| \sum_{\pi \in \mathring{\mathbf{P}}: 0 < h_\pi \lesssim \delta^{500}} h_\pi g_\pi \right\|_q \lesssim \delta^{500} |\pi| \#\mathring{\mathbf{P}} \lesssim \delta^{400}.$$

We group the rectangles π by value of coefficients h_π . Since $\|g\|_{p,2\delta} = 1$, from (3.2) we can see that $h_\pi \lesssim 1$. For any dyadic number $\delta^{500} \lesssim h \lesssim 1$ we define $\mathring{\mathbf{P}}_h := \{\pi \in \mathring{\mathbf{P}} : h \leq h_\pi < 2h\}$. It is classified into $\mathring{\mathbf{P}}_{h,\Theta} := \mathring{\mathbf{P}}_h \cap \mathbf{P}_\Theta$, and let

$$\mathring{\mathbf{P}}_h^k := \bigcup_{k \leq \#\mathring{\mathbf{P}}_{h,\Theta} < 2k} \mathring{\mathbf{P}}_{h,\Theta}$$

for dyadic numbers $1 \leq k \lesssim \delta^{-2}$. Since there are $O(\log \delta^{-1})$ dyadic numbers $\delta^{500} \lesssim h \lesssim 1$ and $1 \leq k \lesssim \delta^{-2}$, by pigeonholing there exist h and k so that

$$\left\| \sum_{\delta^{500} \leq h \lesssim 1} h \sum_{1 \leq k \lesssim \delta^{-2}} \sum_{\pi \in \mathring{\mathbf{P}}_h^k} g_\pi \right\|_q \lesssim (\log \delta^{-1})^2 h \left\| \sum_{\pi \in \mathring{\mathbf{P}}_h^k} g_\pi \right\|_q.$$

Let $\tilde{g} := \sum_{\pi \in \mathring{\mathbf{P}}_h^k} g_\pi$. Then from these estimates, one has

$$\|g\|_q \lesssim \delta^{-\varepsilon} h \|\tilde{g}\|_q + \delta^{400}.$$

Since \tilde{g} is a balanced function, from Höler's inequality, (3.5) and Lemma 3.2 it follows that

$$\|\tilde{g}\|_q \leq \|\tilde{g}\|_{q_1}^{1-\theta} \|\tilde{g}\|_{q_2}^\theta \leq A_1^{1-\theta} A_2^\theta \|\tilde{g}\|_{p_1, 2\delta}^{1-\theta} \|\tilde{g}\|_{p_2, 2\delta}^\theta \lesssim A_1^{1-\theta} A_2^\theta \|\tilde{g}\|_{p, 2\delta},$$

and by (3.2),

$$h \|\tilde{g}\|_{p, 2\delta} \lesssim \|g\|_{p, 2\delta}.$$

Therefore, by combining these estimates we obtain (3.7). \square

To prove Theorem 1.1 we need a trilinear interpolation lemma. Before stating the lemma let us define a notation, which will be repeatedly used in the remaining parts of this paper. We define $\underline{\underline{\prod}}$ by

$$\underline{\underline{\prod}} A_i := \left(\prod_{i=1}^3 |A_i| \right)^{1/3}.$$

From simple calculations it is easy to see the followings.

$$\begin{aligned} \underline{\underline{\prod}} A &= A, \\ \underline{\underline{\prod}} C A_i &= C \underline{\underline{\prod}} A_i, \\ \underline{\underline{\prod}} (A_i B_i) &= \underline{\underline{\prod}} A_i \underline{\underline{\prod}} B_i, \\ \underline{\underline{\prod}} A_i^\alpha &= \left(\underline{\underline{\prod}} A_i \right)^\alpha \quad \text{for } \alpha \in \mathbb{R}. \end{aligned}$$

Also, by Hölder's inequality it follows that for $1 \leq p \leq \infty$,

$$\left(\sum_{\Delta} \underline{\underline{\prod}} A_{i, \Delta}^p \right)^{1/p} \leq \underline{\underline{\prod}} \left(\sum_{\Delta} A_{i, \Delta}^p \right)^{1/p}, \quad (3.8)$$

$$\left\| \underline{\underline{\prod}} f_i \right\|_p \leq \underline{\underline{\prod}} \|f_i\|_p. \quad (3.9)$$

Now we state our trilinear interpolation lemma.

Lemma 3.4. *Assume that*

$$\left\| \underline{\underline{\prod}} f_i \right\|_{q_1} \leq A_1 \underline{\underline{\prod}} \|f_i\|_{p_1, \delta}, \quad \left\| \underline{\underline{\prod}} f_i \right\|_{q_2} \leq A_2 \underline{\underline{\prod}} \|f_i\|_{p_2, \delta} \quad (3.10)$$

for all f_i , $i = 1, 2, 3$, with $\hat{f}_i \subset \Gamma_\delta$. Suppose that for some $\theta \in (0, 1)$,

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$$

and $2 \leq p \leq q \leq \infty$. Then

$$\left\| \underline{\underline{\prod}} f_i \right\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta \underline{\underline{\prod}} \|f_i\|_{p, \delta}$$

for all f_i , $i = 1, 2, 3$, with $\hat{f}_i \subset \Gamma_\delta$ and all $\varepsilon > 0$.

Proof. The proof is similar to Lemma 3.3. We decompose $\underline{\underline{\prod}} f_i = \sum_{k \in \delta^{-1}\mathbb{Z}^3} \psi_k \underline{\underline{\prod}} f_i$ where $\psi_k := \psi(\delta(x - k))$. We reduced it with analogous method in the proof of Lemma 3.3. By localization, it suffices to show that

$$\left\| \underline{\underline{\prod}} g_i \right\|_q \lesssim \delta^{-\varepsilon} A_1^{1-\theta} A_2^\theta \quad (3.11)$$

for all g_i with $\hat{f}_i \subset \Gamma_\delta$ and $\|g_i\|_{p,2\delta} = 1$. Some minor portions can be removed as in the proof in Lemma 3.3. Since ψ_0 has fast decay outside $B(0, C\delta^{-1})$, we have $\|\prod g_i\|_q \leq \|\prod g_i\|_{L^q(B(0, \delta^{-1-\varepsilon}))} + C_K \delta^K$ for all $\varepsilon > 0$ and $K > 0$. Since g_i is Fourier supported in $\Gamma_{2\delta}$, by Lemma 3.1,

$$g_i(x) = \sum_{\Theta_i \in \Pi_\delta} \sum_{\pi_i \in \mathbf{P}_{\Theta_i}} h_{\pi_i} g_{\pi_i}(x).$$

By (3.4), we can eliminate π_i that is disjoint from $B(0, C\delta^{-1-\varepsilon})$, so we can restrict \mathbf{P}_i to the collection $\mathring{\mathbf{P}}_i$ of π_i intersecting $B(0, C\delta^{-1-\varepsilon})$. We can also remove π_i with $0 < h_{\pi_i} \lesssim \delta^{500}$.

For $\delta^{500} \lesssim h_i \lesssim 1$, we define $\mathring{\mathbf{P}}_{h_i} := \{\pi \in \mathring{\mathbf{P}}_i : h_i \leq h_\pi < 2h_i\}$. Let $\mathring{\mathbf{P}}_{\Theta_i}(h_i) := \mathring{\mathbf{P}}_{h_i} \cap \mathbf{P}_{\Theta_i}$, and for any dyadic number $1 \leq k \lesssim \delta^{-2}$ we define

$$\mathring{\mathbf{P}}_{h_i}^{k_i} = \bigcup_{k_i \leq \#\mathring{\mathbf{P}}_{\Theta_i}(h_i) < 2k_i} \mathring{\mathbf{P}}_{\Theta_i}(h_i).$$

Then, we have

$$\|\prod g_i\|_q \lesssim \|\prod \left(\sum_{\delta^{500} \lesssim h_i \lesssim 1} h_i \sum_{1 \lesssim k_i \lesssim \delta^{-2}} \sum_{\pi \in \mathring{\mathbf{P}}_i(h_i, k_i)} g_{\pi_i} \right)\|_q + \delta^{100}.$$

We write as

$$\prod_{i=1}^3 \left(\sum_{h_i} h_i \sum_{k_i} \sum_{\pi \in \mathring{\mathbf{P}}_i(h_i, k_i)} g_{\pi_i} \right) = \sum_{h_1, h_2, h_3} \sum_{k_1, k_2, k_3} \prod_{i=1}^3 \left(h_i \sum_{\pi \in \mathring{\mathbf{P}}_i(h_i, k_i)} g_{\pi_i} \right).$$

By dyadic pigeonholing, there exist dyadic numbers h_i and k_i , $i = 1, 2, 3$, so that

$$\|\prod \left(\sum_{\delta^{500} \lesssim h_i \lesssim 1} h_i \sum_{k_i} \sum_{\pi \in \mathring{\mathbf{P}}_i(h_i, k_i)} g_{\pi_i} \right)\|_q \lesssim (\log \delta^{-1})^2 \prod h_i \|\prod \left(\sum_{\pi \in \mathring{\mathbf{P}}_i(h_i, k_i)} g_{\pi_i} \right)\|_q.$$

Let $\tilde{g}_i := \sum_{\pi \in \mathring{\mathbf{P}}_i(h_i, k_i)} g_{\pi_i}$. Then from these estimates we have

$$\|\prod g_i\|_q \lesssim \delta^{-\varepsilon} \prod h_i \|\prod \tilde{g}_i\|_q + \delta^{100}.$$

Since \tilde{g}_i are balanced functions, from Höler's inequality, (3.10) and Lemma 3.2 it follows that

$$\|\prod \tilde{g}_i\|_q \leq \|\prod \tilde{g}_i\|_{q_1}^{1-\theta} \|\prod \tilde{g}_i\|_{q_2}^\theta \leq A_1^{1-\theta} A_2^\theta \prod \|\tilde{g}_i\|_{p_1, 2\delta}^{1-\theta} \prod \|g_i\|_{p_2, 2\delta}^\theta \lesssim A_1^{1-\theta} A_2^\theta \delta^{-\varepsilon} \prod \|\tilde{g}_i\|_{p, 2\delta}$$

and by (3.2),

$$h_i \|\tilde{g}_i\|_{p, 2\delta} \lesssim \|g_i\|_{p, 2\delta}.$$

Therefore, these estimates yield (3.11). \square

4. PROOF OF THEOREM 1.1.

This section is devoted to the proof of $\mathcal{SQ}(4 \rightarrow 4; 1/16)$. By Proposition 2.2 this follows from the trilinear square function estimate $\mathcal{SQ}(4 \times 4 \times 4 \rightarrow 4; 1/16)$. To prove this we will utilize the following two theorems. The first one is the multilinear restriction theorem due to Bennet, Carbery and Tao [1].

Theorem 4.1. *Let f_i , $i = 1, 2, 3$, be supported in Γ^{Ω_i} . Suppose that $\Gamma^{\Omega_1}, \Gamma^{\Omega_2}, \Gamma^{\Omega_3}$ are ν -transverse. If $R \gg \nu^{-1}$ then for any $\epsilon > 0$ and any ball Q_R of radius $R > 0$,*

$$\left\| \prod \widehat{f_j d\sigma} \right\|_{L^3(Q_R)} \leq C_\epsilon R^\epsilon \prod \|f_j\|_2. \quad (4.1)$$

The second one is the l^2 decoupling theorem due to Bourgain and Demeter [3].

Theorem 4.2. *Suppose that the Fourier support of f is contained in Γ_δ . Then for any $\epsilon > 0$,*

$$\|f\|_6 \leq C_\epsilon \delta^{-\epsilon} \left(\sum_{\Theta \in \Pi_\delta} \|f_\Theta\|_6^2 \right)^{1/2}. \quad (4.2)$$

To deal with local estimates we define local norms as follows:

$$\|f\|_{L^p(\psi_B)} := \|f\psi_B\|_p.$$

and for any functions f with $\text{supp } \hat{f} \subset \Gamma_\delta$,

$$\|f\|_{p,\delta,B} := \left(\sum_{\Theta \in \Pi_\delta} \|f_\Theta\|_{L^p(\psi_B)}^2 \right)^{1/2}.$$

Note that if B is a ball of radius $\geq 2/\sqrt{\delta}$ then for $p \geq 2$,

$$\|f\psi_B\|_{p,\delta} \lesssim \|f\|_{p,\delta,B}. \quad (4.3)$$

Indeed, we decompose the Fourier transform of $(f\psi_B) * \Xi_\Theta$ as follows:

$$(\hat{f} * \hat{\psi}_B) \hat{\Xi}_\Theta = ((\hat{f} \hat{\Xi}_{C\Theta}) * \hat{\psi}_B) \hat{\Xi}_\Theta + ((\hat{f}(1 - \hat{\Xi}_{C\Theta})) * \hat{\psi}_B) \hat{\Xi}_\Theta.$$

Consider the last term of the above equation. We write as

$$((\hat{f}(1 - \hat{\Xi}_{C\Theta})) * \hat{\psi}_B)(x) \hat{\Xi}_\Theta(x) = \int \hat{f}(y)(1 - \hat{\Xi}_{C\Theta})(y) \hat{\psi}_B(x - y) \hat{\Xi}_\Theta(x) dy.$$

From $y \in \Gamma_\delta \setminus C\Theta$ and $x \in \Theta$ we have $|x - y| \geq \sqrt{\delta}$. Since $\hat{\psi}_B$ is supported in a ball of radius $\leq \sqrt{\delta}/2$ with center 0, the above equation is zero. Thus, by Fourier inversion,

$$(f\psi_B) * \Xi_\Theta = ((f * \Xi_{C\Theta})\psi_B) * \Xi_\Theta.$$

By this equation and Young's inequality, we have

$$\|(f\psi_B) * \Xi_\Theta\|_p \lesssim \|(f * \Xi_{C\Theta})\psi_B\|_p \lesssim \sum_{\Theta' \subset C\Theta} \|(f * \Xi_{\Theta'})\psi_B\|_p,$$

therefore we have (4.3).

4.1. We will deduce a trilinear decoupling estimate from Theorem 4.1 and Theorem 4.2. By using a localization argument and a slicing argument, from Theorem 4.1 it follows that

$$\left\| \prod f_i \right\|_3 \leq C_\epsilon \delta^{1/2-\epsilon} \prod \|f_i\|_2$$

for all f_i with $\text{supp } \hat{f}_i \subset \Gamma_\delta^{\Omega_i}$, (for the details, see [10]*Lemma 2.2, [14]). By orthogonality, one has that if f is a function with $\text{supp } \hat{f} \subset \Gamma_\delta$, then

$$\|f\|_2 \sim \left(\sum_{\Theta \in \Pi_\delta} \|f_\Theta\|_2^2 \right)^{1/2} = \|f\|_{2,\delta}.$$

Thus we have

$$\left\| \prod f_i \right\|_3 \leq C_\epsilon \delta^{1/2-\epsilon} \prod \|f_i\|_{2,\delta}.$$

On the other hand, from (4.2) using Hölder's inequality we have

$$\left\| \prod f_i \right\|_6 \leq C_\epsilon \delta^{-\epsilon} \prod \|f_i\|_{6,\delta}.$$

We interpolate these two estimates by using Lemma 3.4. Then,

$$\left\| \prod f_i \right\|_4 \leq C_\epsilon \delta^{1/4-\epsilon} \prod \|f_i\|_{3,\delta}.$$

By Hölder's inequality one has $\|f_i\|_{3,\delta} \leq \|f_i\|_{4,\delta}^{2/3} \|f_i\|_{2,\delta}^{1/3}$. Inserting this into the above we obtain

$$\left\| \prod f_i \right\|_4 \leq C_\epsilon \delta^{1/4-\epsilon} \left(\prod \|f_i\|_{4,\delta} \right)^{2/3} \left(\prod \|f_i\|_{2,\delta} \right)^{1/3}. \quad (4.4)$$

4.2. Set $R = \delta^{-1}$. We take a covering $\{\Delta\}$ of \mathbb{R}^3 by finite overlapped $2R^{1/2}$ -balls. We apply the estimate (4.4) to $f_i\psi_\Delta$. Since the Fourier support of $f_i\psi_\Delta$ is in $\Gamma_{2\sqrt{\delta}}$, by (4.4) and (4.3) we obtain

$$\left\| \prod f_i \right\|_{L^4(\Delta)} \leq C_\epsilon \sqrt{R}^{-1/4+\epsilon} \left(\prod \|f_i\|_{4,\sqrt{\delta},\Delta} \right)^{2/3} \left(\prod \|f_i\|_{2,\sqrt{\delta},\Delta} \right)^{1/3}.$$

After taking 4th power to the above, we sum over Δ , and apply Hölder's inequality. Then,

$$\sum_\Delta \left\| \prod f_i \right\|_{L^4(\Delta)}^4 \leq C_\epsilon \sqrt{R}^{-1+4\epsilon} \left(\sum_\Delta \prod \|f_i\|_{4,\sqrt{\delta},\Delta}^4 \right)^{2/3} \left(\sum_\Delta \prod \|f_i\|_{2,\sqrt{\delta},\Delta}^4 \right)^{1/3}.$$

After taking 4th root to the above, we apply (3.8) to summations in the right-hand side. Then,

$$\begin{aligned} \left(\sum_\Delta \left\| \prod f_i \right\|_{L^4(\Delta)}^4 \right)^{1/4} &\leq C_\epsilon \sqrt{R}^{-1/4+\epsilon} \left(\left(\sum_\Delta \prod \|f_i\|_{4,\sqrt{\delta},\Delta}^4 \right)^{1/4} \right)^{2/3} \left(\left(\sum_\Delta \prod \|f_i\|_{2,\sqrt{\delta},\Delta}^4 \right)^{1/4} \right)^{1/3} \\ &\leq C_\epsilon \sqrt{R}^{-1/4+\epsilon} \left(\prod \left(\sum \|f_i\|_{4,\sqrt{\delta},\Delta}^4 \right)^{1/4} \right)^{2/3} \left(\prod \left(\sum \|f_i\|_{2,\sqrt{\delta},\Delta}^4 \right)^{1/4} \right)^{1/3}. \end{aligned}$$

We have $\left(\sum_\Delta \|f_i\|_{4,\sqrt{\delta},\Delta}^4 \right)^{1/4} \lesssim \|f_i\|_{4,\sqrt{\delta}}$ by Minkowski's inequality. Thus, from the above estimate it follows that

$$\left\| \prod f_i \right\|_4 \leq C_\epsilon \sqrt{R}^{-1/4+\epsilon} \left(\prod A_i \right)^{2/3} \left(\prod B_i \right)^{1/3}, \quad (4.5)$$

where

$$A_i := \|f_i\|_{4,\sqrt{\delta}}, \quad B_i := \left(\sum_\Delta \|f_i\|_{2,\sqrt{\delta},\Delta}^4 \right)^{1/4}.$$

4.3. We will show that

$$B_i \lesssim R^{3/8} \|S_\delta f_i\|_4. \quad (4.6)$$

By definition we write $\|f_i\|_{2,\sqrt{\delta},\Delta}^2 = \sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \|f_{i,\Upsilon}\|_{L^2(\psi_\Delta)}^2$. Since $f_{i,\Upsilon}$ is decomposed as $f_{i,\Upsilon} = \sum_{\Theta \in \Pi_\delta: \Theta \subset 2\Upsilon} f_{i,\Theta}$, we have $\|f_i\|_{2,\sqrt{\delta},\Delta}^2 = \sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \int |\sum_{\Theta \in \Pi_\delta: \Theta \subset 2\Upsilon} f_{i,\Theta} \psi_\Delta|^2$. We see that the Fourier support of $f_{i,\Theta} \psi_\Delta$ is contained in a $C\Theta$, so by orthogonality it follows that

$$\|f_i\|_{2,\sqrt{\delta},\Delta}^2 \lesssim \sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \sum_{\Theta \in \Pi_\delta: \Theta \subset 2\Upsilon} \int |f_{i,\Theta} \psi_\Delta|^2 \lesssim \sum_{\Theta \in \Pi_\delta} \int |f_{i,\Theta} \psi_\Delta|^2.$$

Since $\sum_{\Theta \in \Pi_\delta} \int |f_{i,\Theta} \psi_\Delta|^2 = \int (\sum_{\Theta \in \Pi_\delta} |f_{i,\Theta}|^2)^{\frac{1}{2} \times 2} \psi_\Delta^2 = \|S_\delta f_i\|_{L^2(\psi_\Delta)}^2$, the above estimate is written as

$$\|f_i\|_{2,\sqrt{\delta},\Delta} \lesssim \|S_\delta f_i\|_{L^2(\psi_\Delta)}.$$

By using this estimate and Hölder's inequality,

$$B_i \lesssim \left(\sum_\Delta \|S_\delta f_i\|_{L^2(\psi_\Delta)}^4 \right)^{1/4} \lesssim R^{\frac{3}{2}(\frac{1}{2}-\frac{1}{4})} \left(\sum_\Delta \|S_\delta f_i\|_{L^4(\psi_\Delta)}^4 \right)^{1/4} \lesssim R^{3/8} \|S_\delta f_i\|_4.$$

Thus we obtain (4.6).

4.4. We will use an induction argument. Suppose that $\alpha \geq 0$ is the best constant such that $\mathcal{SQ}(4 \times 4 \times 4 \rightarrow 4; \alpha)$. To prove $\mathcal{SQ}(4 \times 4 \times 4 \rightarrow 4; 1/16)$ it is enough to show that for any $\epsilon > 0$,

$$\alpha \leq \frac{1}{16} + C\epsilon.$$

By Hölder's inequality,

$$A_i \lesssim R^{\frac{1}{4}(\frac{1}{2}-\frac{1}{4})} \left(\sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \|f_{i,\Upsilon}\|_4^4 \right)^{1/4}.$$

By the induction hypothesis and Lemma 2.2 one has $\mathcal{SQ}(4 \rightarrow 4; \alpha)$. By (2.5),

$$\|f_{i,\Upsilon}\|_4 \leq C_\epsilon R^{\alpha/2+\epsilon} \|S_\delta f_{i,\Upsilon}\|_4.$$

So, we have

$$A_i \leq C_\epsilon R^{\alpha/2+\epsilon} R^{\frac{1}{4}(\frac{1}{2}-\frac{1}{4})} \left(\sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \|S_\delta f_{i,\Upsilon}\|_4^4 \right)^{1/4}.$$

Since

$$\begin{aligned} \sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \|S_\delta f_{i,\Upsilon}\|_4^4 &\lesssim \sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \int \left(\sum_{\Theta \in \Pi_\delta: \Theta \subset 2\Upsilon} |f_{i,\Theta}|^2 \right)^2 \\ &\lesssim \int \left(\sum_{\Upsilon \in \Pi_{\sqrt{\delta}}} \sum_{\Theta \in \Pi_\delta: \Theta \subset 2\Upsilon} |f_{i,\Theta}|^2 \right)^2 \\ &\lesssim \|S_\delta f_i\|_4^4, \end{aligned}$$

we obtain

$$A_i \leq C_\epsilon R^{1/16+\alpha/2+\epsilon} \|S f_i\|_4. \quad (4.7)$$

Now we insert (4.7) and (4.6) into (4.5). Then,

$$\left\| \prod f_i \right\|_{L^4(Q_R)} \leq C_\epsilon R^{1/24+\alpha/3+C\epsilon} \prod \|S f_i\|_4.$$

Since α is the best constant holding $\mathcal{SQ}(4 \times 4 \times 4 \rightarrow 4; \alpha)$, we have $\alpha \leq \frac{1}{24} + \frac{\alpha}{3} + C\epsilon$. Therefore, $\alpha \leq \frac{1}{16} + C\epsilon$. This completes the proof.

5. PROOF OF THEOREM 1.3.

In this section, Theorem 1.3 will be proved by using a corresponding trilinear estimate. As usual, by Littlewood-Paley decomposition we may replace $e^{it\sqrt{-\Delta}} f$ with $U_N f$, defined by

$$U_N f(x, t) = \check{\eta}_N * e^{it\sqrt{-\Delta}} f(x),$$

with an ϵ -loss in α , where η_N is a bump function supported in $\{\xi \in \mathbb{R}^2 : |\xi| \sim N\}$ and $\check{\eta}_N$ is the inverse Fourier transform of η_N . For convenience of rescaling we reform $U_N f$ as follows. By a linear transformation $J : (\xi_1, \xi_2, \xi_3) \mapsto (\zeta_1, \zeta_2, \zeta_3) = (\xi_3 - \xi_1, \xi_2, \xi_3 + \xi_1)$ which maps the cone $\{(\xi_1, \xi_2, \pm\sqrt{\xi_1^2 + \xi_2^2})\}$ to a leaned cone $\{(\zeta_1, \zeta_2, \zeta_2^2/\zeta_1)\}$, we redefine $U_N f$ by

$$U_N f(x, t) = \int e^{2\pi i(x \cdot \xi + t\xi_2^2/\xi_1)} \hat{f}(\xi) \eta_N(\xi_1) \varphi(\xi_2/\xi_1) d\xi, \quad \xi = (\xi_1, \xi_2),$$

where φ is a bump function supported in the unit interval. Then, $U_N f$ has Fourier support in

$$\Gamma(N) := \{(\xi_1, \xi_2, \xi_2^2/\xi_1) : |\xi_1| \sim N, |\xi_2/\xi_1| \lesssim 1\}.$$

The leaned cone $(\xi_1, \xi_2, \xi_2^2/\xi_1)$ is written as $\xi_1(1, \theta, \theta^2)$ provided $\theta = \xi_2/\xi_1$. So one may identify θ with an angular variable of the cone.

We say that the local smoothing estimate $\mathcal{LS}(p \rightarrow q; \alpha)$ holds if

$$\|U_N f\|_{L^q(\mathbb{R}^2 \times [1, 2])} \leq C_\epsilon N^{\alpha+\epsilon} \|f\|_p \quad (5.1)$$

holds for all $\epsilon > 0$, all $N \geq 1$ and all $f \in L^p(\mathbb{R}^2)$. To prove Theorem 1.3 it suffices to show

$$\mathcal{LS}(10/3 \rightarrow 30/7; 1/10).$$

As before, we will use an induction method. For this we set a hypothesis as follows. For given $\epsilon > 0$, $1 \leq p < q \leq \infty$ and $\frac{1}{p} + \frac{3}{q} = 1$, we define

$$\alpha = \alpha(p, q) \geq \frac{1}{p} - \frac{3}{q} + \frac{1}{2} \quad (5.2)$$

to be the best exponent for which the estimate (5.1) holds for all $N \geq 1$ and all $f \in L^p(\mathbb{R}^2)$. Then it is enough to show that

$$\alpha\left(\frac{10}{3}, \frac{30}{7}\right) \leq \frac{1}{10} + C_\epsilon + \log_N C_\epsilon. \quad (5.3)$$

5.1. Let another arbitrary small $\epsilon_1 > 0$ be given. Let $N \geq N_0$ and $1 > \gamma_1 > \gamma_2 \geq N_0^{-\epsilon_1/2}$. Later, γ_1, γ_2 and N_0 will be chosen. By rescaling and Lemma 2.1 one has that for any $(x, t) \in \mathbb{R}^2 \times [1, 2]$,

$$\begin{aligned} |U_N f(x, t)| &\lesssim \max_{\Omega \in \Omega(\gamma_1)} |U_N^\Omega f(x, t)| + \gamma_1^{-1} \max_{\Omega \in \Omega(\gamma_2)} |U_N^\Omega f(x, t)| \\ &\quad + \gamma_2^{-50} \max_{\substack{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_2): \\ \text{dist}(\Omega_i, \Omega_j) \geq \gamma_2, i \neq j}} \left| \left(\prod_{i=1}^3 |U_N^{\Omega_i} f(x, t)| \right)^{1/3} \right|, \end{aligned}$$

where

$$U_N^\Omega f(x, t) = \int e^{2\pi i(x \cdot \xi + t\xi_2^2/\xi_1)} \hat{f}(\xi) \eta_N(\xi_1) \varphi_\Omega(\xi_2/\xi_1) d\xi.$$

By embedding $l^q \subset l^\infty$ it follows that

$$\begin{aligned} \|U_N f\|_{L^q(\mathbb{R}^2 \times I)} &\lesssim \left(\sum_{\Omega_1 \in \Omega(\gamma_1)} \|U_N^{\Omega_1} f\|_{L^q(\mathbb{R}^2 \times I)}^q \right)^{1/q} + \gamma_1^{-1} \left(\sum_{\Omega_2 \in \Omega(\gamma_2)} \|U_N^{\Omega_2} f\|_{L^q(\mathbb{R}^2 \times I)}^q \right)^{1/q} \\ &\quad + \gamma_2^{-50} \left(\sum_{\substack{\Omega_1, \Omega_2, \Omega_3 \in \Omega(\gamma_2): \\ \text{dist}(\Omega_i, \Omega_j) \geq \gamma_2, i \neq j}} \left\| \left(\prod_{i=1}^3 |U_N^{\Omega_i} f| \right)^{1/3} \right\|_{L^q(\mathbb{R}^2 \times I)}^q \right)^{1/q}, \end{aligned} \quad (5.4)$$

where $I = [1, 2]$.

We consider the first and second summation in the right-hand side of (5.4). From rescaling and the induction hypothesis it follows that

$$\|U_N^{\Omega_i} f\|_{L^q(\mathbb{R}^2 \times I)} \leq C \gamma_i^{3(\frac{1}{q} - \frac{1}{p})} (\gamma_i^2 N)^{\alpha+\epsilon} \|f\|_p. \quad (5.5)$$

More specifically, by rotating we may assume that Ω is centered at 0. Then we may write $U_N^{\Omega_i} f$ as

$$U_N^{\Omega_i} f(x, t) = \int e^{2\pi i(x \cdot \xi + t\xi_2^2/\xi_1)} \hat{f}(\xi) \eta_N(\xi_1) \varphi(\gamma_i^{-1} \xi_2/\xi_1) d\xi.$$

Let $\sigma(x_1, x_2, t) = (\gamma_i^2 x_1, \gamma_i x_2, t)$ and $\underline{\sigma}(x_1, x_2) = (\gamma_i^2 x_1, \gamma_i x_2)$. Then, we have $U_N^{\Omega_i} f \circ \sigma = U_{\gamma_i^2 N}^{\Omega_i}(f \circ \underline{\sigma})$. Thus, using (5.1) and this relation we have (5.5).

If we define f_Ω by

$$\widehat{f}_\Omega(\xi_1, \xi_2) = \widehat{f}(\xi_1, \xi_2) \chi_{\{|\xi_1| \sim N\}}(\xi_1) \chi_\Omega(\xi_2/\xi_1)$$

where χ denotes a characteristic function, then we may replace $U_N^{\Omega_i} f$ with $U_N^{\Omega_i} f_{\Omega_i}$. By (5.5),

$$\left(\sum_{\Omega_i \in \Omega(\gamma_i)} \|U_N^{\Omega_i} f_{\Omega_i}\|_q^q \right)^{1/q} \leq C \gamma_i^{3(\frac{1}{q} - \frac{1}{p})} (\gamma_i^2 N)^{\alpha+\epsilon} \left(\sum_{\Omega_i \in \Omega(\gamma_i)} \|f_{\Omega_i}\|_p^q \right)^{1/q}.$$

After embedding $l^p \subset l^q$ in the right-hand side of the above, we apply Lemma 5.1 below. Then we obtain

$$\left(\sum_{\Omega_i \in \Omega(\gamma_i)} \|U_N^{\Omega_i} f\|_{L^q(\mathbb{R}^2 \times I)}^q \right)^{1/q} \leq C \gamma_i^{3(\frac{1}{q} - \frac{1}{p})} (\gamma_i^2 N)^{\alpha+\epsilon} \|f\|_p. \quad (5.6)$$

Lemma 5.1 ([13], Lemma 7.1). *Let R_k be a collection of rectangles such that the dilates $2R_k$ are almost disjoint, and suppose that f_k are a collection of functions whose Fourier transforms are supported on R_k . Then for all $1 \leq p \leq \infty$ we have*

$$\left(\sum_k \|f_k\|_p^{p^*} \right)^{1/p^*} \lesssim \left\| \sum_k f_k \right\|_p \lesssim \left(\sum_k \|f_k\|_p^{p^*} \right)^{1/p^*}$$

where $p_* = \min(p, p')$, $p^* = \max(p, p')$.

5.2. We consider the last summation in the right-hand side of (5.4). We will show that for any $\epsilon > 0$,

$$\left\| \prod U_N^{\Omega_i} f \right\|_{L^{30/7}(\mathbb{R}^2 \times I)} \leq C_\epsilon N^{1/10+\epsilon} \|f\|_{10/3}. \quad (5.7)$$

First we prove a corresponding local estimate.

Lemma 5.2. *Let B be a unit ball. Then, for any $\epsilon > 0$,*

$$\left\| \prod |U_N^{\Omega_i} f_i| \right\|_{L^{30/7}(B \times I)} \leq C_\epsilon N^{1/10+\epsilon} \prod \|f_i\|_{10/3}. \quad (5.8)$$

Proof. By interpolation it suffices to show

$$\left\| \prod U_N^{\Omega_i} f_i \right\|_{L^6(B \times I)} \leq C_\epsilon N^{1/6+\epsilon} \prod \|f_i\|_6, \quad (5.9)$$

$$\left\| \prod U_N^{\Omega_i} f_i \right\|_{L^3(B \times I)} \leq C_\epsilon N^\epsilon \prod \|f_i\|_2. \quad (5.10)$$

Consider (5.9). By Hölder's inequality it is enough to show

$$\|U_N f\|_{L^6(B \times I)} \leq C_\epsilon N^{1/6+\epsilon} \|f\|_6. \quad (5.11)$$

Since $\psi_I(t)U_N f(x, t)$ has Fourier support in a C -neighborhood of $\Gamma(N)$, from Theorem 4.2 and rescaling it follows that

$$\|U_N f\|_{L^6(B \times I)} \leq C_\epsilon N^\epsilon \left(\sum_{\tilde{\Theta}} \|(\psi_I U_N f) * \Xi_{\tilde{\Theta}}\|_6^2 \right)^{1/2},$$

where $\tilde{\Theta}$ is a sector of size $CN^{1/2} \times CN \times C$. By Hölder's inequality, this is bounded by

$$\leq C_\epsilon N^{1/6+\epsilon} \left(\sum_{\tilde{\Theta}} \|(\psi_I U_N f) * \Xi_{\tilde{\Theta}}\|_6^6 \right)^{1/6}.$$

It is well known (see [15], Lemma 6.1) that for $p \geq 2$,

$$\left(\sum_{\tilde{\Theta}} \|(\psi_I U_N f) * \Xi_{\tilde{\Theta}}\|_p^p \right)^{1/p} \lesssim \|f\|_p.$$

Thus, we obtain (5.11)

Consider (5.10). In (4.1), the restriction operator $\widehat{f d\sigma}$ can be replaced with $U_1 \check{f}$ where \check{f} denotes the inverse Fourier transform of f . Thus, from Theorem 4.1 and Plancherel's theorem it follows that

$$\left\| \prod U_1^{\Omega_i} f_i \right\|_{L^3(Q_N)} \leq C_\epsilon N^\epsilon \prod \|f_i\|_2.$$

If $s(x, t) = N^{-1}(x, t)$ and $\underline{s}(x) = N^{-1}x$, then $U_N^\Omega f \circ s = U_1^\Omega (f \circ \underline{s})$. So, by changing variables and translation invariance, the above estimate gives (5.10). \square

We now prove that (5.8) implies (5.7). This immediately follows from the next localization lemma.

Lemma 5.3. *Suppose that the local estimate*

$$\left\| \prod U_N^{\Omega_i} f_i \right\|_{L^q(B \times I)} \leq A(N) \prod \|f_i\|_p \quad (5.12)$$

holds for all unit cubes B and all $f_i \in L^p(\mathbb{R}^2)$. If $p \leq q$ then

$$\left\| \prod U_N^{\Omega_i} f_i \right\|_{L^q(\mathbb{R}^2 \times I)} \leq C A(N) \prod \|f_i\|_p \quad (5.13)$$

holds for all $f_i \in L^p(\mathbb{R}^2)$.

Proof. We write as

$$U_N f(x, t) = (K_N(t) * f)(x)$$

where

$$K_N(t)(x) = K_N(x, t) := \int e^{2\pi i(x \cdot \xi + t \xi_2^2 / \xi_1)} \eta_N(\xi_1) \varphi(\xi_2 / \xi_1) d\xi.$$

By using a stationary phase method, it follows that

$$|K_N(t)(x)| \leq C_M N^2 (1 + N|x|)^{-M} \quad \forall M > 0.$$

Thus, for $(x, t) \in I \times \mathbb{R}^2$,

$$|U_N f(x, t)| \leq C_M (a_N * |f|)(x), \quad \forall M > 0, \quad (5.14)$$

where $a_N(x) = N^2(1 + N|x|)^{-M}$. This estimate will be used for error estimation.

If a unit lattice cube $B \subset \mathbb{R}^2$ is given, then we decompose

$$|U_N f|_{\chi_{B \times I}} \lesssim |U_N(f \chi_{3B})|_{\chi_{B \times I}} + C_M |\mathcal{E}_{B^c} f|_{\chi_{B \times I}}, \quad (5.15)$$

where

$$\mathcal{E}_{B^c} f := a_N * (|f| \chi_{\mathbb{R}^2 \setminus 3B}).$$

By dyadic decomposing,

$$\left(\sum_B \|\mathcal{E}_{B^c} f\|_{L^q(B)}^q \right)^{1/q} \lesssim \left(\sum_B \left\| \sum_{k=1}^{\infty} a_N * (|f| \chi_{2^{k+1}B \setminus 2^k B}) \right\|_{L^q(B)}^q \right)^{1/q}.$$

Since $a_N(x-y) \lesssim N^2(N2^k)^{-M}$ whenever $|x-y| \gtrsim 2^k$, we have

$$\begin{aligned} \chi_B(x)(a_N * (|f|\chi_{2^{k+1}B \setminus 2^k B}))(x) &= \chi_B(x) \int a_N(x-y)\chi_{2^{k+1}B \setminus 2^k B}(y)|f(y)|dy \\ &\lesssim N(2^k N)^{-M/2} \chi_B(x) \int a_N^{1/2}(x-y)|f(y)|dy \\ &\lesssim N^{-M/2+1} 2^{-kM/2} \chi_B(x)(a_N^{1/2} * |f|)(x). \end{aligned}$$

By inserting this into the previous inequality,

$$\left(\sum_B \|\mathcal{E}_{B^c} f\|_{L^q(B)}^q \right)^{1/q} \lesssim N^{-M/2+1} \sum_k 2^{-kM/2} \|a_N^{1/2} * |f|\|_q.$$

Thus, by Young's inequality we obtain

$$\left(\sum_B \|\mathcal{E}_{B^c} f\|_{L^q(B)}^q \right)^{1/q} \lesssim N^{-M/2+C} \|f\|_p \quad (5.16)$$

for sufficiently large M .

On the other hand, by some rough estimates we see that $\|U_N f\|_{L^q(B \times I)} \lesssim N^C \|f\|_p$. So, by embedding $l^p \subset l^q$, we have

$$\left(\sum_B \|U_N(f\chi_{2B})\|_{L^q(B \times I)}^q \right)^{1/q} \lesssim N^C \left(\sum_B \|f\|_{L^p(2B)}^q \right)^{1/q} \lesssim N^C \|f\|_p. \quad (5.17)$$

Now, with (5.16) and (5.17) we consider the estimate (5.13). We define f_{Ω_i} as

$$\widehat{f_{\Omega_i}}(\xi_1, \xi_2) = \widehat{f_i}(\xi) \eta_N(\xi_1) \varphi_{\Omega_i}(\xi_2/\xi_1).$$

Then we may replace $U_N^{\Omega_i} f_i$ with $U_N f_{\Omega_i}$. By (5.15),

$$\begin{aligned} \prod U_N f_{\Omega_i} \chi_{B \times I} &\lesssim \prod \left(|U_N(f_{\Omega_i} \chi_{3B})| \chi_{B \times I} + C_M (\mathcal{E}_{B^c} f_{\Omega_i}) \chi_{B \times I} \right) \\ &\lesssim \prod |U_N(f_{\Omega_i} \chi_{3B})| \chi_{B \times I} + C_M \mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3}) \chi_{B \times I}, \end{aligned} \quad (5.18)$$

where

$$\begin{aligned} \mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3}) &:= \sum_{i,j,k} (\mathcal{E}_{B^c} f_{\Omega_i} |U_N(f_{\Omega_j} \chi_{2B})| |U_N(f_{\Omega_k} \chi_{2B})|)^{1/3} \\ &\quad + \sum_{i,j,k} (\mathcal{E}_{B^c} f_{\Omega_i} \mathcal{E}_{B^c} f_{\Omega_j} |U_N(f_{\Omega_k} \chi_{2B})|)^{1/3} + \prod \mathcal{E}_{B^c} f_{\Omega_i}. \end{aligned}$$

By Minkowski's inequality,

$$\begin{aligned} &\left(\sum_B \|\mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3})\|_{L^q(B \times I)}^q \right)^{1/q} \\ &\lesssim \left(\sum_B \|(\mathcal{E}_{B^c} f_{\Omega_i} |U_N(f_{\Omega_j} \chi_{2B})| |U_N(f_{\Omega_k} \chi_{2B})|)^{1/3}\|_{L^q(B \times I)}^q \right)^{1/q} \\ &\quad + \left(\sum_B \|(\mathcal{E}_{B^c} f_{\Omega_i} \mathcal{E}_{B^c} f_{\Omega_j} |U_N(f_{\Omega_k} \chi_{2B})|)^{1/3}\|_{L^q(B \times I)}^q \right)^{1/q} \\ &\quad + \left(\sum_B \left\| \prod \mathcal{E}_{B^c} f_{\Omega_i} \right\|_{L^q(B \times I)}^q \right)^{1/q}. \end{aligned} \quad (5.19)$$

Consider the right-hand side of (5.19). By Hölder's inequality,

$$\begin{aligned} & \left(\sum_B \|(\mathcal{E}_{B^c} f_{\Omega_i} | U_N(f_{\Omega_j} \chi_{2B}) | U_N(f_{\Omega_k} \chi_{2B}))\|^{1/3} \|_{L^q(B \times I)}^q \right)^{1/q} \\ & \leq \left(\sum_B \|\mathcal{E}_{B^c} f_{\Omega_i}\|_{L^q(B \times I)}^q \right)^{1/3q} \left(\sum_B \|U_N(f_{\Omega_j} \chi_{2B})\|_{L^q(B \times I)}^q \right)^{1/3q} \\ & \quad \times \left(\sum_B \|U_N(f_{\Omega_k} \chi_{2B})\|_{L^q(B \times I)}^q \right)^{1/3q}. \end{aligned}$$

Thus, by (5.16) and (5.17),

$$\left(\sum_B \|(\mathcal{E}_{B^c} f_{\Omega_i} | U_N(f_{\Omega_j} \chi_{2B}) | U_N(f_{\Omega_k} \chi_{2B}))\|^{1/3} \|_{L^q(B \times I)}^q \right)^{1/q} \lesssim N^{-M/6+C} \prod \|f_{\Omega_i}\|_p.$$

The second and third summations in the right-hand side of (5.19) are estimated by an analogous method. Thus,

$$\left(\sum_B \|\mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3})\|_{L^q(B \times I)}^q \right)^{1/q} \lesssim N^{-M/6+C} \prod \|f_{\Omega_i}\|_p. \quad (5.20)$$

By (5.18)

$$\begin{aligned} \|\prod U_N f_{\Omega_i}\|_{L^q(\mathbb{R}^2 \times I)} &= \left(\sum_B \|\prod U_N f_{\Omega_i}\|_{L^q(B \times I)}^q \right)^{1/q} \\ &\lesssim \left(\sum_B \|\prod U_N(f_{\Omega_i} \chi_{2B})\|_{L^q(B \times I)}^q \right)^{1/q} + \left(\sum_B \|\mathcal{E}(f_{\Omega_1}, f_{\Omega_2}, f_{\Omega_3})\|_{L^q(B \times I)}^q \right)^{1/q}. \end{aligned}$$

By (5.12), (5.20) and embedding $l^p \subset l^q$, it follows that

$$\|\prod U_N f_{\Omega_i}\|_{L^q(\mathbb{R}^2 \times I)} \lesssim (A(N) + N^{-M/6+C}) \prod \|f_{\Omega_i}\|_p.$$

Since $\|f_{\Omega_i}\|_p \lesssim \|f_i\|_p$, we obtain (5.13) provided a sufficiently large M is taken. \square

5.3. Last of all, we will show (5.3). We substitute (5.6) and (5.7) in (5.4) with $(p, q) = (10/3, 30/7)$. Then, it follows that

$$\|U_N f\|_{L^{30/7}(I \times \mathbb{R}^2)} \lesssim (\gamma_1^{2\alpha - \frac{1}{5} + 2\epsilon} N^{\alpha + \epsilon} + \gamma_1^{-1} \gamma_2^{2\alpha - \frac{1}{5} + 2\epsilon} N^{\alpha + \epsilon} + C_\epsilon \gamma_2^{-60} N^{\frac{1}{10} + \epsilon}) \|f\|_{10/3}. \quad (5.21)$$

So, by the assumption that α is a best exponent,

$$N^\alpha \leq C(\gamma_1^{2\alpha - \frac{1}{5} + 2\epsilon} + \gamma_1^{-1} \gamma_2^{2\alpha - \frac{1}{5} + 2\epsilon}) N^\alpha + C_\epsilon \gamma_2^{-60} N^{\frac{1}{10}}.$$

Observe that $2\alpha - \frac{1}{5} \geq 0$ by (5.2). We now choose γ_1, γ_2 and N_0 so that $C\gamma_1^{2\alpha - \frac{1}{5} + 2\epsilon} \leq 1/4$, $C\gamma_1^{-1} \gamma_2^{2\alpha - \frac{1}{5} + 2\epsilon} \leq 1/4$ and $1 > \gamma_1 > \gamma_2 \geq N_0^{-\epsilon_1/2}$. Then $N^\alpha \leq C_{\epsilon_1, \epsilon} N^{\frac{1}{10} + 60\epsilon_1}$. Since ϵ_1 is arbitrary, we take $\epsilon_1 = \epsilon$. Then we obtain (5.3).

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